Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

TEACHING PUPILS HOW TO STUDY MATHEMATICS

By ALFRED DAVIS
Soldan High School, St. Louis, Mo.

II.

So far the suggestions which concern study and teaching have been of a general nature, applicable, for the most part, to any study. We shall now discuss in greater detail the study of geometry. We select this subject because it offers unusual opportunities to both teacher and student for both teaching and study. The suggestions made for geometry will make clear the proper methods of procedure for other mathematical studies. Moreover, the student who handles geometry successfully is probably equipped to study any subject to which he wishes to apply himself.

Some teachers start a class in geometry by spending the first few days in the reviewing of algebra. This is a serious mistake. It matters not how expert the pupil may be in algebra, nor how well he may like the subject, such a procedure at the beginning is a distinct loss. The pupil comes with an appetite for a new subject, and he is curious to know what that subject will be like. The wise teacher will use this advantage to begin the study of geometry at once. The teacher who fails to do this raises unnecessary barriers to the pupil's interest and success—a disadvantage that, in some cases, may never be overcome. If the review of algebra should be necessary, let it come at the time the subject is needed in the geometry.

The work in geometry should not begin with formal demonstrations. To plunge into this is to bewilder and to discourage the pupil. He cannot recall anything like these proofs in his experience; and so it is impossible for him to make the proper associations whereby the new material can be related to what is already known. Accordingly, for a long time (perhaps for all time) the study means nothing to the student. It does not help matters for the teacher to tell the pupil that he subject will clear up later, the chances are that the prospect for this has been already lost in the dislike of the student for the subject. It is usually such procedure at the beginning of the course that leads pupils to say, "Geometry is hard," or "I hate geometry." The difficulties of a pupil of normal ability are more than likely due to mistakes in teaching.

In this day of enlightenment and of progress it may be safe to assume that no teacher will begin geometry by assigning long lists of definitions and axioms to be memorized without any apparent reason therefore. These should be learned only when needed for further progress in the study of the subject. At such times it is convenient to refer to them as tools with which to work, just as the carpenter needs to know the use of the hammer, saw, etc., before he can build anything worth while, so we must be thoroughly familiar with certain tools before we can build our interesting proofs. Since these definitions, which are the fundamental hypotheses for our work, are justified by experience, we must see to it that the experience of the pupil is extended, if necessary, to enable him to thoroughly understand their meaning and to feel justified in accepting them. It often happens that what is perfectly obvious to the teacher, is "Greek" to the pupil. A constant effort must be made to ascertain what the pupil knows, for it is only as we build on this as a foundation that our efforts can have any value whatever.

The introduction of a class to the study of geometry is one of the most important parts of the course. Success or failure frequently depends on this. The subject should be made interesting from the beginning. The teacher should question the class as to what their notions of geometry and its values are. With this as his starting point he should lead them to a general survey of the ground to be covered, and the advantages to be sought in the study. The subject should be given an historical setting. The part it has played in great engineering efforts, its importance in further progress, and the invincibility of its conclusions will appeal to the pupil. The table of contents in the text may be used as a guide for the student when giving a survey of the progress of the study.

Having stirred the pupil's interest and expectation the pupil should be provided with some simple instruments such as ruler, compass, protractor, etc. Such exercises as the following should be worked out neatly by the pupil under suitable direction:

Bisect a line.

Bisect an angle.

Draw a perpendicular to a line at a point on the line, at a point not on the line, at the end of the line.

Circumscribe a circle about a triangle.

Inscribe a circle in a triangle.

Inscribe a hexagon, a square, etc., in a given circle.

Make some interesting designs, some of which may be devised by the pupil.

It will be desirable to have the pupil put these constructions neatly in a note-book. This note-book may later contain the most difficult and most interesting exercises of the course, and such matter as the pupil needs for reference, as well as exercises and supplementary work assigned from time to time by the teacher.

This work will acquaint the pupil with geometric forms, constructions, definitions, etc.; develop accuracy that will be important in later work; and above all it will maintain interest, enthusiasm, and appreciation. Formal proofs of these exercises should not be attempted at this time. However, the pupil should check his work by measurement, and when he can he may, in an informal way, say why he thinks the construction ought to be accurate. This will lead, in the case of some constructions, to a realization for the need of a proof, and the pupil will have an appetite for the more formal proof to come later. The minimum time for this sort of work is about a week; the work should not be allowed to drag; if interest cannot be maintained in one thing try something else, keeping in mind the inexperience and limitations of pupil.

It is the utmost folly to begin our proofs by trying to prove something that seems perfectly obvious to the pupil. The pupil must realize the need for a proof, or the work will be trivial and unimportant for him. Those propositions which the experience of the pupil justifies him in accepting should be postulated—assumed true. It may be that he will later see need for the proofs and then they may be taken up again. The following scheme has been tried with success. The teacher goes over the statements of the theorems carefully with the class. If every member of the class is willing to accept the theorem without proof it is indicated on the blackboard in the column for theorems not to be proved; if all think, or if any member of the class thinks, a proof is needed, the theorem is indicated in the column for those needing proof. As the game progresses the class may transfer some that they at first thought did not need proof to the other column. All those which finally remain in the assumed class are assumed, at least for the time being. Of course, the success of such an experiment depends on the skill of the teacher. The pupil is likely to assume the truth of such theorems as: All right angles are equal. All straight angles are equal, etc. If the pupil is inclined to assume too many theorems, a few exercises may be used to convince him that looking at a figure is not always sufficient evidence for an assumption. Considerable interest may be aroused by having a number of pupils who have seen the same event report on the details of the occurrence, and noting the contradictions that come up. Two parallel lines may be drawn on the board, and these crossed by numerous other lines drawn through a point midway between them. For most pupils the lines will now seem bent and no longer parallel. Other devices to be found in any modern geometry text may be used. The pupils are usually surprised and interested with these experiments and will be led to question some of the statements which seemed obvious before. Again, if the teacher wishes to start with a theorem which the class does not see the need of proving, it may be studied as a model of what a proof ought to be, and as an easy example of this. If two straight lines intersect, the vertically opposite angles are equal is such a theorem. Indeed all the theorems worked out in the book should be treated as models, and the exercises should be treated as theorems upon which the pupil is to apply the methods learned from the models and so measure his own progress. Otherwise, all the proved theorems of the text might be learned without mastering geometry.

The necessary parts of a proof may be approached and illustrated by reference to debates, with which the pupil is probably somewhat familiar. In this way he will learn the meaning of a proof in geometry, and he will early appreciate its clear, concise, and accurate statements, and the certainty of its conclusions. He will rapidly acquire confidence in his own ability to think. The study of geometry should be directly related to the study of debate, not only by the teacher of geometry, but also by the teacher of debating.

It is usually a good plan to require only informal and oral proofs during the first weeks. To insist on completely written proofs from the beginning is to make the work unnecessarily difficult, and to discourage the pupil. Time is needed to become acquainted with this new method of reasoning, and it should be our first concern to have the pupil feel comfortably at home with the process. The pupil's attention may, at the proper stage, be directed to the importance of being able to write his arguments in a convincing manner. A few attempts at this, followed by a thorough and constructive discussion will lead to the mastery of the formal proof. The teacher may aid the pupil greatly by working out the proof of an "original", with the class and by writing it on the board. This exercise may then be erased from the board and assigned for home work. If the results are satisfactory other "originals" may be assigned to be worked out by the pupil without assistance. It would, however, be drudgery to require that all originals be written out in this laborious manner. Such a requirement would rob the best part of geometry of much of its interest. Great care should be exercised by the teacher in the selection of problems. No text has yet been so well devised that all the problems it contains should be worked by any one class, nor does any text contain all the problems that a teacher may use to advantage. Reasonable care should be given to accuracy in the drawing of the figure used, and to the form of the written proof. It is not always desirable, however, to insist on a set form of written work for every member of the class. The class must be taught to analyze a problem for the purpose of discovering proofs for themselves.

The writer uses the following method with much success: After the class has become somewhat accustomed to the solving of exercises, anticipate a new theorem by giving the statement of it to the class, but keep the fact that it is another proved theorem a secret. Discuss the new problem and by skilful questioning, where necessary, develop the proof. When the work has been completed the class will be surprised and elated to learn that they have proved the next theorem, and it will usually seem an easy matter, besides teaching concretely how to study a geometry lesson. The pupils will read the proof for this theorem as given in the text with great interest to discover if possible any variation from their own proof. If any differences can be found the pupils can be impressed with the fact that they are studying a department of knowledge and not merely a text. They will read the text more carefully thereafter, and try to find proofs other than those given. Of course pupils should be encouraged to find new proofs. They should be told if this is necessary, that such work counts more to their credit than any amount of reproducing of what is given in the text. The method just suggested is applicable early in the work and for especially difficult theorems as well as for all exercises that give trouble. It will not do to anticipate every advance lesson, or the interest of the class, and its motive for effort, will be destroyed. The pleasure from the study of geometry consists in the mastery of difficult problems without help from anyone, and in learning how to gain this mastery.

The interest of the pupils may be stimulated by the practical application of geometry where this is convenient. The pupil may be assigned the making of simple instruments, and these may be used in working out problems in the classroom or out-of-doors. For example, the pupil may make a quadrant and use this to measure the height of objects, etc. Valuable suggestions along this line may be found in the Stone-Millis Plane Geometry, 1916 edition, and in other modern texts. It must be remembered that the purpose of this work is chiefly to stimulate interest. It can be overdone. It will be hard, for example, to persuade the intelligent pupil that he is really learning how to do the work of the world. In most cases the student, knowing of the instruments of precision used in making measurements, will soon grow weary of using his own crude instruments, and it is not always advisable to provide expensive instruments, this for obvious reasons. Again, it is not easy for the average teacher to keep a large group interested in a problem out-of-doors, even though a detailed report be required. Anything more than the occasional field trip for work out-of-doors is a waste of time and effort.

Interest may be further sustained by referring to the number of proofs for the Theorem of Pythagoras, especially to one by President Garfield, and others recently discovered by high school students. Some of these may be found in back numbers of *School Science and Mathematics,* and in other periodicals.

The pupils will be interested in the three famous problems of antiquity, and they will be eager to go to encyclopedias and other sources for information about these problems. Some member of the class will be interested in constructing an instrument for trisecting an angle. Newell and Harper's Geometry may be used by the pupil for suggestions. They will be interested in seeing the value of Pi given to 707 places as it is in *Mathematical Wrinkles,* by S. I. Jones.

The attention of the pupils may be called to some of the great problems of geometry and of mathematics. For example: the discovery of the planet Neptune by mathematics; the location of big guns in the recent war; calculating distances of bodies remote from the earth; all of these to show the pupils how important geometry has been in the progress of civilization. Take them to see great bridges, buildings, great engineering achievements within reach, and show the important part mathematics had in their construction. The list of interesting things related to the study of geometry could be made endless. Those named are representative of what the alert teacher, who has a passion for his subject, may do. Under such guidance some pupils will be inspired with a lofe for the study of mathematics and determine to study the subject further because of the service it will enable them to render their fellows. They will be impressed with the fact that the world is only on the threshold of progress and that going further depends in large measure on the use of mathematics. Under proper instruction pupils will not lack motive for the study of mathematics.

The following are a few suggestions to be placed in the hands of the student or to be given orally to him by the teacher, on

''How to Study a Theorem Which Is Proved in the Text.''

Their puprose is to aid the student in mastering the art of study, especially as applied to geometry. It would be an excellent idea to have such suggestions incorporated in the text to be used by the pupil, or placed in the pupil's hands in the form of a hand book on how to study mathematics.

The pupil should first ask himself, "What are the purposes to be aimed at and accomplished as far as possible, in the study of this theorem?" There are three important purposes, namely:

1. To learn the best way of studying, and of thinking out a problem.
2. To be able to use the theorem in proving other principles in geometry.
3. To use the theorem in solving problems encountered in life.

Which of these three purposes is of greatest concern to you? Is not the first by far the most important? Keeping this in mind will enable you to guard against the formation of improper habits of study. The formation of correct habits in this study will be of great assistance to you in your other studies. This study has advantages over many other studies in cultivating your ability to concentrate your mind on a problem and to ignore distractions. You can also be sure of your reasoning because you can test your results.

Memorizing the Proof. Since the most important thing connected with the theorem is the manner in which you study it, do not attempt to memorize the proof given in the text. That is, do not learn the steps by heart, for such would be mental slavery. Rather get the proof by some more thoughtful method. Learn to think for yourself and become independent of the book.

Attempt to Find a Proof of Your Own First. Do not permit yourself to look at the figure or proof in the text until you have first attempted to find a proof of your own. The author probably proved the theorem in full because he thought it difficult for you. He got all the fun which such work gives. The real pleasure is in doing it yourself. You may sometimes fail to find the proof yourself. But even then, as you will readily see, you have an advantage in the fact that you are able to understand and appreciate the author's proof better by comparison with the steps in your own effort. It is always exceedingly interesting and valuable to see how someone else easily accomplished a thing—like working a puzzle, for example—which you had attempted time after time with failure. The important points of the author's proof which you failed to see are more thoroughly impressed on your mind.

In attempting to find a proof of your own before looking at that in the text, determine carefully the hypothesis and conclusion; and be sure that you understand positively the meanings of all the terms involved. How do you find the meaning of a term when you do not know it? If you clearly understand the theorem you should be able to picture it clearly in your mind.

Draw a figure of your own, using your own lettering. Make the figure general. That is, if the theorem speaks of *any* triangle, do not draw a *right* triangle, nor an *isosceles* triangle. For you might be led by the special figure into a mistake in reasoning. Make the drawing accurate, because an accurately drawn figure will often suggest the proof. If it is at all convenient, use compass and straightedge in drawing the figure. Thus, if the figure is supposed to contain a perpendicular to a line, actually construct the perpendicular with the ruler and compass.

Now in endeavoring to discover the proof, it is usually best to begin by considering the conclusion

1. What kind of relation does the conclusion involve? Proving lines equal? Proving lines parallel? Proving angles equal? Proving a proportion? If it is proving lines equal, recall the different ways in which this is done—by congruence of triangles, opposite sides of a parellelogram, etc.

2. Recall any previous theorems, corollaries, or definitions thus related to the conclusion, on which the proof may be made to depend. Continue until the required links for connecting the hypothesis and conclusion are found.

3. If necessary, try to draw auxiliary lines that will help in the proof.

4. If you can find no direct proof in this way, think whether or not the indirect, or analytic, method of proof may be used. That is, proceed step by step from the conclusion to the hypothesis, asking at each point ho wthis result can be justified.

Studying the Proof in the Text. If you succeed in discovering a proof of your own, then read that of the text and compare the two. If you fail to discover a proof, study the author's proof. Observe especially the critical points of the latter which you failed to think of in your own struggle. These are your weaknesses, and attention to them in this particular connection will impress them thoroughly on your mind.

Looking Up the References. The text does not usually give the reasons in full for the steps, but gives references in the margin to sections where these reasons may be found. The purpose of this is not merely to save space, but to give you opportunities to think for yourself. Hence, in studying the author's proof, do not look up a reference unless you cannot recall it. Test your accuracy by looking up the reference after you have decided what it is. Do this with especial care if there is any doubt in your mind. If you are sure you know the reference do not take time to look it up.

In reciting the proof orally, or in writing it, you should state in full the reasons for all steps. Do not refer to them by number, as that would place the useless burden on the mind of memorizing the number attached to each reason.

Getting the Main Points of the Proof. In studying the proof of the book, try to see the big ideas involved. Is the proof direct or indirect? Is the main idea congruence of triangles? Is it proportion? Is it a principle in parallelograms? Is it a comparison of angles? Is it superposition of figures?

Determine the most important steps of the proof. Pick out the critical points, or the turning points, of the proof. If these steps are mastered the others become minor details which are easily filled in.

To what other theorems is the present theorem most closely related?

It is this organization of the proof—getting the big ideas, picking out the most important steps, or turning points of the proof, and finding the most important relationships of the theorem to other theorems—that should be striven for.

Writing the Proof. After the proof of the theorem has been grasped it should be carefully written out without reference to the book. Your accuracy may be tested, if there is any doubt, by comparison with that of the text. Nothing will fix the proof more securely in mind than writing it out.

Applying the Theorem. Your knowledge of the theorem is not complete until you can use it, either in proving other principles of geometry, or in solving problems in practical situations.

Hence, examine the exercises in the book, especially those following closely after the theorem, to determine the use to be made of the theorem. Usually a number of such exercises can easily be found. Try to find out how to use the theorem in solving these exercises.

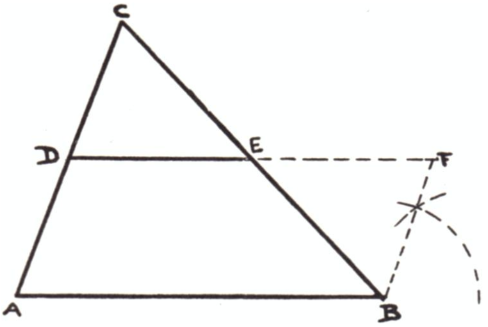
Be on the alert to find opportunity to use the theorem in some practical situation in your own work. Think of problems outside of the text in which the theorem may have application.

An Example. Let us illustrate the method outlined above by applying it to the study of the following theorem taken from the 1916 edition of the Stone-Millis text, page 76:

The line segment which joins the middle points of two sides of a triangle is parallel to the third side and is equal to one-half of it.

You will remember that the thing you value most is the learning how to study; hence, you will try to discover a proof for the theorem before looking at the one in the book.

Having first gotten the meaning of the theorem clearly in mind, so that you can imagine the figure, you will draw accurately with instruments the following figure to represent the theorem.



You will formulate:

(a) Hypothesis—In triangle *ABC, D* is the middle point of *AC,* and *E* the middle point of *BC.*

(b) Conclusion—$DE \parallel AB$, and $DE = \frac{1}{2}AB$.

In attacking the proof, you see that you are to prove lines parallel and line segments equal. There comes to your mind several methods of accomplishing each of these. You weigh them: Lines are proved parallel by means of a transversal or by use of a parallelogram. Since you know nothing here about the angles formed by a transversal, you decide to use a parallelogram. Again, line-segments are possibly most often proved equal by congruence of triangles or by means of a parallelogram. The parallelogram method is chosen. Hence, in order to form a parallelogram, you produce *DE* through *E*, and draw $BF \parallel AC$ to meet *DE* at *F*.

You may be able to finish the proof or you may fail. In either case, you finally come to read the proof in the book which is as follows:

1. *D* and *E* are middle points of *AC* and *BC*, respectively, *i. e.*, $AD = DC$ and $BE = EC$. Hyp.

2. Produce *DE* through *E*, and draw $BF \parallel AC$, meeting *DE* produced at *F*.

3. $\angle DEC = \angle BEF$. Verticle $\angle$s.

4. $\angle DCE = \angle EBF$. Sec. 29.

5. $\therefore$ Triangle *DEC* is congruent with triangle *BEF*. Sec. 65.

6. $\therefore BF = DC$ and $EF = DE$. Def. of Congruence.

7. $\therefore BF = AD$. Ax. 1

8. $\therefore$ *ABFD* is a parallelogram. Sec. 90.

9. $\therefore DE \parallel AB$. Def. of Parallelogram.

10. Also $DF = AB$. Sec. 82.

11. $\therefore DE = \frac{1}{2} AB$. Ax. 5.

In each step try to think for yourself what theorem, corollary, or definition is the reason referred to. You should be able to recall it; but if not, then, and not before, is the time to look it up.

It is evident that you will be completely lost if you have not reviewed from day to day the theorems, etc., you have previously studied so that they may come to your mind when you need them.

Next get the big thought of the proof. What is it? Is it not the parallelogram? Is this the only idea of great importance? No, another is the congruence of triangles in proving *ABFD* a parallelogram. Which, then, are the important steps of the proof? Steps 5 and 8. All other steps are of less importance, and relate themselves to these turning points of the proof. Fix in mind, then, the movement of the proof, from the congruence of triangles to the parallelogram.

When you are sure that you thoroughly understand the proof write it out in full and then compare what you have written with the text.

Study the proof to find other ways of handling it. If you find another way be prepared to offer this in class. This will not only give you pleasure and satisfaction, and improve your standing in the class; but it will be an incentive to others to do the same sort of work. For example, in this theorem we might have produced DE to F making $EF = DE$, and have drawn FB. The triangles could then be proved congruent because two sides and the included angle of one would be equal respectively to the two sides and the included angle of the other.

Look among the theorems, corollaries, and exercises that follow this theorem to find some to which it applies. You will find the following:

The line-segment connecting the middle points of the non-parallel sides of a trapezoid is parallel to the bases and equal to one-half the sum of the bases.

The line-segments connecting the middle points of the adjacent sides of any quadrilateral form a parallelogram.

The line-segments connecting the middle points of the opposite sides of any quadrilateral bisect each other.

If $ABCD$ is a parallelogram, and E and F are the middle points of DC and AB, respectively; prove that AE and CF bisect DB.

The medians of a triangle are concurrent at a point of trisection of each (two proofs).

It will also help us to establish the fact that the middle point of the hypotenuse of a right triangle is equally distant from the three vertices; and that if the hypotenuse is double the shorter leg the acute angls are 30° and 60°.

Try to prove these exercises, and others you may be able to find. It may be necessary to wait until other facts are proved before you can prove some of these exercises. Make a note of such and be on the look out for the needed information as the work proceeds.

What is the converse of this theorem? Do you find any proof for this?

Determine the relative value of this theorem as compared with other theorems, and be prepared to give reasons in the class for your decision. Compare your classification of the theorem with that of other members of the class. Note carefully any suggestions the teacher may make in this connection.

Keep a note book and record all the interesting constructions not given in the text, which have come up in the work. Keep a careful record of all facts, sidelights, and applications of the theorems proved, and indicate carefully the place in the text to which these apply.

If the spirit of these directions is followed, as well as the letter, any student can master geometry and obtain from its study the benefits to be derived from it.

To summarize briefly, as a student you should:

1. Review what has preceded so as to insure its recall when needed.
2. Read the statement of the theorem.
3. Without reference to the proof in the text try to work out a proof of your own, remembering that you have in hand all the information needed for the proof.
4. Compare your proof with that of the book.
5. If you do not succeed, or if you have made a mistake, master the points in which you have failed, with the determination to succeed the next time.
6. Be sure to identify the main, or critical, points of the proof. When these are mastered the rest is easy.
7. Try to recall all references to reasons. If you are not sure look them up again making an effort to remember them permanently.
8. It is usually advisable to write out the theorem with the text closed, and then compare what you have written with the text.
9. Relate the theorem to facts already studied, and apply it to exercises.

(To be continued)